

LEFT APP-RINGS OF SKEW GENERALIZED POWER SERIES *

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Abstract

A ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s -unital as an ideal of R for any $a \in R$. Let R be a ring, (S, \leq) a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. The skew generalized power series ring $[[R^{S, \leq}, \omega]]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Malcev-Neumann Laurent series rings. We study the left APP-property of the skew generalized power series ring $[[R^{S, \leq}, \omega]]$. It is shown that if (S, \leq) is a strictly totally ordered monoid, $\omega : S \rightarrow \text{Aut}(R)$ a monoid homomorphism and R a ring satisfying descending chain condition on right annihilators, then $[[R^{S, \leq}, \omega]]$ is left APP if and only if for any S -indexed subset A of R , the ideal $l_R(\sum_{a \in A} \sum_{s \in S} R\omega_s(a))$ is right s -unital.

Key Words: left APP-ring, skew generalized power series ring.

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1. Introduction and preliminaries

Throughout this paper, R denotes a ring (not necessarily commutative) with unity. For a nonempty subset X of R , $l_R(X)$ and $r_R(X)$ denote the left and right annihilator of X in R , respectively. We will denote by $\text{End}(R)$ the monoid of ring endomorphisms of R , and by $\text{Aut}(R)$ the group of ring automorphisms of R .

Recall that a ring R is a right (resp. left) PP-ring if the right (resp. left) annihilator of an element of R is generated by an idempotent. The ring R is called a PP-ring if it is both right and left PP. A ring R is called (quasi-) Baer if the left annihilator of every nonempty subset (every left ideal) of R is generated by an idempotent of R . For more details and examples of PP-rings, Baer rings and

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quasi-Baer rings, see [2, 3, 5, 7, 8, 10]. As a generalization of quasi-Baer rings, G.F. Birkenmeier, J.Y. Kim and J.K. Park in [6] introduced the concept of left principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply, left p.q.-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right p.q.-Baer rings can be defined. A ring R is called p.q.-Baer if it is both right and left p.q.-Baer. Observe that biregular rings and quasi-Baer rings are left p.q.-Baer. According to [1], a commutative ring R is called PF-ring if the annihilator $\text{ann}_R(a)$ is pure as an ideal of R for every $a \in R$. As a common generalization of left p.q.-Baer rings, right PP-rings and PF-rings, the concept of left APP-rings was introduced in [18]. A ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is pure as a left ideal of R for every $a \in R$. For more details and examples of left APP-rings, see [18] and [9].

There are a lot of results concerning left principal quasi-Baerness, right PP-property and PF-property of polynomial extensions and power series extensions of a ring. We recall some of them as follows. It was proved in [4, Theorem 2.1] that a ring R is left p.q.-Baer if and only if $R[x]$ is left p.q.-Baer. C.Y. Hong, N.K. Kim and T.K. Kwak showed in [10, Corollary 15] that if σ is a rigid endomorphism of R , then R is a left p.q.-Baer ring if and only if $R[x; \sigma, \delta]$ is a left p.q.-Baer ring. G.F. Birkenmeier and J.K. Park in [7, Theorem 1.2] showed that if M is a u.p.-monoid, then $R[M]$ is left p.q.-Baer if and only if R is left p.q.-Baer. For skew monoid rings it was proved in [21, Theorem 5] that if M is an ordered monoid and $\phi : M \rightarrow \text{Aut}(R)$ a monoid homomorphism, then the skew monoid ring $R * M$ induced by the monoid homomorphism ϕ is a left p.q.-Baer ring if and only if $l_R(\sum_{g \in M} R\phi(g)(a))$ is generated by an idempotent of R for every $a \in R$. If all right semicentral idempotents of R are central, then it was proved in [15, Theorem 3] that $R[[x]]$ is left p.q.-Baer if and only if R is left p.q.-Baer and every countable family of idempotents of R has a generalized join in the set of all idempotents of R . It was proved in [2, Theorem A] that R is a reduced PP-ring if and only if $R[x]$ is a reduced PP-ring. J. Fraser and W.K. Nicholson in [8] showed that $R[[x]]$ is a reduced PP-ring if and only if R is a reduced PP-ring and every countable family of idempotents of R has a least upper bound in the set of all idempotents of R . It was proved in [1] that $R[[x]]$ is a PF-ring if and only if for any two countable subsets A and B of R with $A \subseteq \text{ann}_R(B)$, there exists $r \in \text{ann}_R(B)$ such that $ar = a$ for all $a \in A$.

In recent years, many researches have carried out an extensive study of rings of (skew) generalized power series (for example, P. Ribenboim[26, 27, 28], Z.K. Liu[14, 16, 17, 19], H. Kim[11, 12], R. Mazurek and M. Ziembowski[22, 23, 24, 25], and the present author [31, 32], etc.). In particular, it was shown in [32, Corollary 3.8] that if (S, \leq) is a strictly totally ordered monoid and R a ring satisfying the condition that $ab = 0 \iff a\omega_s(b) = 0$ for any $a, b \in R$ and any $s \in S$, then $[[R^{S, \leq}, \omega]]$ is a left p.q.-Baer ring if and only if for any S -indexed set A of R , $l_R(\sum_{a \in A} Ra)$ is generated by an idempotent of R . In [31, Corollary 5.5], we proved that $[[R^{S, \leq}]]$ is a reduced PP-ring if and only if R is a reduced PP-ring and for every S -indexed subset A of idempotents of R , $\text{ann}_R(A)$ is generated by an idempotent of R if and only if R is a reduced PP-ring and any S -indexed subset of idempotents of R has

a least upper bound in the set of all idempotents of R . H. Kim and T.I. Kwon proved in [12, Theorem 2.4] that if (S, \leq) is a strictly totally ordered monoid, then $[[R^{S, \leq}]]$ is a PF-ring if and only if for any two S -indexed subsets A and B of R with $A \subseteq \text{ann}_R(B)$, there exists $r \in \text{ann}_R(B)$ such that $ar = a$ for all $a \in A$.

For left APP-rings, it was proved in [21, Theorem 2] that if M is an ordered monoid and $\phi : M \rightarrow \text{Aut}(R)$ is a monoid homomorphism, then the skew monoid ring $R * M$ is a left APP-ring if and only if for any $b \in R$, $l_R(\sum_{g \in M} R\phi(g)(b))$ is pure as a left ideal of R . It was noted in [18, Example 2.4] that there exists a commutative von Neumann regular ring R (hence left APP), but the ring $R[[x]]$ is not APP. In [20, Theorem 2], it was shown that if R is a ring satisfying descending chain condition on right annihilators then $R[[x, \alpha]]$ is a left APP-ring if and only if for any sequence (b_0, b_1, \dots) of elements of R the ideal $l_R(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} R\alpha^k(b_j))$ is pure as a left ideal of R , where $\alpha \in \text{Aut}(R)$.

In this note, we will consider left APP-property of skew generalized power series rings. We will show that if (S, \leq) is a strictly totally ordered monoid, $\omega : S \rightarrow \text{Aut}(R)$ a monoid homomorphism and R is a ring satisfying descending chain condition on right annihilators, then $[[R^{S, \leq}, \omega]]$ is left APP if and only if for any S -indexed subset A of R , the ideal $l_R(\sum_{a \in A} \sum_{s \in S} R\omega_s(a))$ is pure as a left ideal of R .

In order to recall the skew generalized power series ring construction, we need some definitions. Let (S, \leq) be a partially ordered set. Recalled that (S, \leq) is artinian if every strictly decreasing sequence of elements of S is finite, and that (S, \leq) is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [28],[19] and [25].

Let R be a ring, (S, \leq) a strictly ordered monoid (that is, (S, \leq) is an ordered monoid such that if $s, s', t \in S$ and $s < s'$, then $s+t < s'+t$), and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. For any $s \in S$, let ω_s denote the image of s under ω , that is $\omega_s = \omega(s)$. Consider the set A of all maps $f : S \rightarrow R$ whose support $\text{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$$

is finite. This fact allows to define the operation of convolution as follows:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v)), \quad \text{if } X_s(f, g) \neq \emptyset$$

and $(fg)(s) = 0$ if $X_s(f, g) = \emptyset$. With this operation and pointwise addition, A becomes a ring, which is called the ring of skew generalized power series with coefficients in R and exponents in S , and we denote by $[[R^{S, \leq}, \omega]]$.

The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Malcev-Neumann Laurent series rings and of course the "untwisted" versions of all of these.

If (S, \leq) is a strictly totally ordered monoid and $0 \neq f \in [[R^{S, \leq}, \omega]]$, then $\text{supp}(f)$ is a nonempty well-ordered subset of S . We denote $\pi(f)$ the smallest element of $\text{supp}(f)$. To any $r \in R$ and any $s \in S$ we associated the maps $\lambda_r^s \in [[R^{S, \leq}, \omega]]$ defined by

$$\lambda_r^s(t) = \begin{cases} r, & t = s, \\ 0, & t \neq s, \end{cases} \quad t \in S.$$

In particular, denote $c_r = \lambda_r^0$, $e_s = \lambda_1^s$. It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S, \leq}, \omega]]$, $s \mapsto e_s$ is a monoid embedding of S into the multiplicative monoid of ring $[[R^{S, \leq}, \omega]]$, and $\lambda_r^s = c_r e_s$, $e_s c_r = c_{\omega_s(r)} e_s$.

2. Main Results

An ideal I of R is said to be right s -unital if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. Note that if I and J are right s -unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$). It follows from [30, Theorem 1] that I is right s -unital if and only if for any finitely many elements $a_1, a_2, \dots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x, i = 1, 2, \dots, n$. A submodule N of a left R -module M is called a pure submodule if $L \otimes_R N \longrightarrow L \otimes_R M$ is a monomorphism for every right R -module L . By [29, Proposition 11.3.13], an ideal I is right s -unital if and only if R/I is flat as a left R -module if and only if I is pure as a left ideal of R .

By [18], a ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s -unital as an ideal of R for any element $a \in R$.

Right APP-rings may be defined analogously. Clearly every left p.q.-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings). If R is a commutative ring, then R is APP if and only if R is FP. From [18, Proposition 2.3] it follows that right PP-rings are left APP and left APP-rings are quasi-Armendariz in the sense that whenever $f(x) = a_0 + a_1 x + \dots + a_m x^m$, $g(x) = b_0 + b_1 x + \dots + b_n x^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_i R b_j = 0$ for each i and j (see, for example [9]). For more details on left APP-rings, see [18, 9].

Lemma 1. *Let (S, \leq) be a strictly totally ordered monoid and $\omega : S \longrightarrow \text{Aut}(R)$ a monoid homomorphism. If $l_R(\sum_{s \in S} R\omega_s(a))$ is right s -unital for any $a \in R$, then for any $f, g \in [[R^{S, \leq}, \omega]]$ satisfy $g[[R^{S, \leq}, \omega]]f = 0$, $g(u)\omega_u(R\omega_s(f(v))) = 0$ for any $u, v, s \in S$.*

Proof. Let $0 \neq f, g \in [[R^{S, \leq}, \omega]]$ be such that $g[[R^{S, \leq}, \omega]]f = 0$. Assume that $\pi(g) = u_0$ and $\pi(f) = v_0$. Then for any $(u, v) \in X_{u_0+v_0}(g, f)$, $u_0 \leq u$, $v_0 \leq v$. If $u_0 < u$, since \leq is a strict order, $u_0 + v_0 < u + v_0 \leq u + v = u_0 + v_0$, a contradiction. Thus $u = u_0$. Similarly, $v = v_0$. Hence, for any $r \in R$ and any $s \in S$,

$$\begin{aligned} 0 &= (g\lambda_r^s f)(u_0 + s + v_0) = \sum_{(u, v) \in X_{u_0+s+v_0}(g, \lambda_r^s f)} g(u)\omega_u(r\omega_s(f(v))) \\ &= g(u_0)\omega_{u_0}(r\omega_s(f(v_0))). \end{aligned}$$

Now let $w \in S$ with $u_0 + v_0 \leq w$. Assume that for any $u \in \text{supp}(g)$ and any $v \in \text{supp}(f)$, if $u + v < w$, then $g(u)\omega_u(R\omega_s(f(v))) = 0$ for any $s \in S$. We will show

that $g(u)\omega_u(R\omega_s(f(v))) = 0$ for any $s \in S$, any $u \in \text{supp}(\phi)$ and any $v \in \text{supp}(f)$ with $u + v = w$. For convenience, we write

$$X_w(g, f) = \{(u_i, v_i) \mid i = 1, 2, \dots, n\}$$

with $v_1 < v_2 < \dots < v_n$ (Note that if $v_1 = v_2$, then from $u_1 + v_1 = u_2 + v_2$ it follows that $u_1 = u_2$, and thus $(u_1, v_1) = (u_2, v_2)$). Then for any $r \in R$ and any $s \in S$,

$$\begin{aligned} 0 &= (g\lambda_r^s f)(s + w) = \sum_{(u,v) \in X_w(g, \lambda_r^s f)} g(u)\omega_u(r\omega_s(f(v))) \\ &= \sum_{i=1}^n g(u_i)\omega_{u_i}(r\omega_s(f(v_i))). \end{aligned} \quad (1)$$

Note that $u_i + v_1 < u_i + v_i = w$ for each $i = 2, \dots, n$. Then by induction hypothesis, $g(u_i)\omega_{u_i}(R\omega_t(f(v_1))) = 0$ for any $t \in S$ and each $i = 2, \dots, n$. Thus $\omega_{u_i}^{-1}(g(u_i)) \in l_R(\sum_{t \in S} R\omega_t(f(v_1)))$ since $\omega_{u_i} \in \text{Aut}(R)$ for any $i = 2, \dots, n$. Hence there exists $e_1 \in l_R(\sum_{t \in S} R\omega_t(f(v_1)))$ such that $g(u_i) = g(u_i)\omega_{u_i}(e_1)$ for $i = 2, \dots, n$ by the hypothesis. Let $r' \in R$, take $r = e_1 r'$ in the equation (1), we have

$$0 = \sum_{i=1}^n g(u_i)\omega_{u_i}(e_1 r' \omega_s(f(v_i))) = \sum_{i=2}^n g(u_i)\omega_{u_i}(r' \omega_s(f(v_i))). \quad (2)$$

Since $u_i + v_2 < u_i + v_i = w$ for any $i = 3, \dots, n$, by hypothesis, there exists $e_2 \in l_R(\sum_{t \in S} R\omega_t(f(v_2)))$ such that $g(u_i) = g(u_i)\omega_{u_i}(e_2)$ for $i = 3, \dots, n$. Hence take $r' = e_2 r''$ in (2) where $r'' \in R$, we deduced that

$$\sum_{i=3}^n g(u_i)\omega_{u_i}(r'' \omega_s(f(v_i))) = 0.$$

Continuing in this manner yields that $g(u_n)\omega_{u_n}(R\omega_s(f(v_n))) = 0$ for any $s \in S$. Consequently, for any $s \in S$,

$$g(u_{n-1})\omega_{u_{n-1}}(R\omega_s(f(v_{n-1}))) = 0, \dots, g(u_1)\omega_{u_1}(R\omega_s(f(v_1))) = 0.$$

Therefore, by transfinite induction, we have shown that $g(u)\omega_u(R\omega_s(f(v))) = 0$ for any $u, v, s \in S$. \square

Lemma 2. Let (S, \leq) be a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. If $[[R^{S, \leq}, \omega]]$ is a left APP-ring and S is cancellative, then $l_R(\sum_{s \in S} R\omega_s(a))$ is right s -unital for any $a \in R$.

Proof. Let $a \in R$ and $b \in l_R(\sum_{s \in S} R\omega_s(a))$. Then $c_b[[R^{S, \leq}, \omega]]c_a = 0$. Since $[[R^{S, \leq}, \omega]]$ is left APP, there exists an $h \in l_{[[R^{S, \leq}, \omega]]}([R^{S, \leq}, \omega]c_a)$ such that $c_b = c_b h$. Then $b = c_b(0) = (c_b h)(0) = b h(0)$ and, for any $r \in R$, any $s \in S$,

$$0 = (h\lambda_r^s c_a)(s) = h(0)r\omega_s(a),$$

which imply that $l_R(\sum_{s \in S} R\omega_s(a))$ is right s -unital for any $a \in R$. \square

Let (S, \leq) be a strictly ordered monoid and A a nonempty subset of R . We will say A is S -indexed, if there exists an artinian and narrow subset I of S such that A is indexed by I .

Theorem 3. *Let (S, \leq) be a strictly totally ordered monoid and $\omega : S \longrightarrow \text{Aut}(R)$ a monoid homomorphism. If R satisfies descending chain condition on right annihilators, then the following conditions are equivalent:*

- (1) $[[R^{S, \leq}, \omega]]$ is a left APP-ring.
- (2) For any S -indexed subset A of R , $l_R(\sum_{a \in A} \sum_{s \in S} R\omega_s(a))$ is right s -unital.

Proof. (2) \implies (1). Assume that $f, g \in [[R^{S, \leq}, \omega]]$ are such that $g[[R^{S, \leq}, \omega]]f = 0$. Then, by the hypothesis and Lemma 1, $g(u)\omega_u(R\omega_s(f(v))) = 0$ for any $u, v, s \in S$. Since $\omega_u \in \text{Aut}(R)$, $\omega_u^{-1}(g(u))R\omega_s(f(v)) = 0$ for any $u, v, s \in S$. Thus for any $u \in \text{supp}(g)$,

$$\omega_u^{-1}(g(u)) \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right).$$

Let

$$\mathcal{D} = \{r_R(Y) | Y \subseteq \{\omega_u^{-1}(g(u)) | u \in \text{supp}(g)\}, |Y| < \infty\}.$$

Then \mathcal{D} is a nonempty set of right annihilators. Since R satisfies descending chain condition on right annihilators, \mathcal{D} has a minimal element, say $r_R(Y_0)$. Assume that $Y_0 = \{\omega_{u_1}^{-1}(g(u_1)), \omega_{u_2}^{-1}(g(u_2)), \dots, \omega_{u_n}^{-1}(g(u_n))\}$. Then

$$\omega_{u_i}^{-1}(g(u_i)) \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right), \quad i = 1, 2, \dots, n.$$

Thus, by (2), there exists $e \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right)$ such that

$$\omega_{u_i}^{-1}(g(u_i)) = \omega_{u_i}^{-1}(g(u_i))e, \quad i = 1, 2, \dots, n.$$

If $\text{supp}(g) = \{u_1, u_2, \dots, u_n\}$, then for all $u \in \text{supp}(g)$, $\omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e$. Now assume that $u \in \text{supp}(g) \setminus \{u_1, u_2, \dots, u_n\}$. Then, by the minimality of $r_R(Y_0)$,

$$r_R(\omega_{u_1}^{-1}(g(u_1)), \dots, \omega_{u_n}^{-1}(g(u_n)), \omega_u^{-1}(g(u))) = r_R(\omega_{u_1}^{-1}(g(u_1)), \dots, \omega_{u_n}^{-1}(g(u_n))).$$

Thus $\omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e$. This implies that $\omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e$ for any $u \in \text{supp}(g)$. Thus for any $h \in [[R^{S, \leq}, \omega]]$ and any $t \in S$,

$$(c_e h f)(t) = \sum_{(s, v) \in X_t(h, f)} e h(s) \omega_s(f(v)) = 0,$$

and

$$(g c_e)(t) = g(t) \omega_t(e) = \omega_t(\omega_t^{-1}(g(t))e) = \omega_t(\omega_t^{-1}(g(t))) = g(t),$$

which imply that $c_e \in l_{[[R^{S, \leq}, \omega]]}([R^{S, \leq}, \omega]f)$ and $g = g c_e$. Hence $[[R^{S, \leq}, \omega]]$ is a left APP-ring.

(1) \implies (2). Let $A = \{a_t | t \in I\}$ be an S -indexed subset of R . Define $f \in [[R^{S, \leq}, \omega]]$ via

$$f(t) = \begin{cases} a_t, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Let $b \in l_R(\sum_{t \in I} \sum_{s \in S} R\omega_s(a_t))$. Then $c_b[[R^{S, \leq}, \omega]]f = 0$. Since $[[R^{S, \leq}, \omega]]$ is left APP, there exists an $h \in l_{[[R^{S, \leq}, \omega]]}([R^{S, \leq}, \omega]]f)$ such that $c_b = c_b h$. Thus $b = c_b(0) = (c_b h)(0) = b h(0)$. By (1), Lemma 2 and Lemma 1, $h(u)\omega_u(R\omega_s(f(t))) = 0$ for any $u, s, t \in S$. In particular, $h(0)R\omega_s(f(t)) = 0$ for any $s, t \in S$. This implies that $h(0) \in l_R(\sum_{t \in I} \sum_{s \in S} R\omega_s(f(t)))$. Thus (2) holds. \square

Corollary 4. ([20, Theorem 2]) *Let R be a ring satisfying descending chain condition on right annihilators and $\alpha \in \text{Aut}(R)$. Then the following conditions are equivalent:*

- (1) $R[[x; \alpha]]$ is a left APP-ring.
- (2) For any countable subset A of R , $l_R(\sum_{a \in A} \sum_{i=0}^{\infty} R\alpha^i(a))$ is right s -unital.

Corollary 5. *Let R be a ring satisfying descending chain condition on right annihilators and $\alpha \in \text{Aut}(R)$. Then the following conditions are equivalent:*

- (1) $R[[x, x^{-1}; \alpha]]$ is a left APP-ring.
- (2) For any countable subset A of R , $l_R(\sum_{a \in A} \sum_{i=-\infty}^{\infty} R\alpha^i(a))$ is right s -unital.

Let α and β be ring automorphisms of R such that $\alpha\beta = \beta\alpha$. Let $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ (resp. $\mathbb{Z} \times \mathbb{Z}$) be endowed the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$ (resp. \mathbb{Z}), and define $\omega : S \longrightarrow \text{Aut}(R)$ via $\omega(m, n) = \alpha^m \beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$ (resp. $m, n \in \mathbb{Z}$). Then $[[R^{S, \leq}, \omega]] = R[[x, y; \alpha, \beta]]$ (resp. $R[[x, y, x^{-1}, y^{-1}; \alpha, \beta]]$), in which $(ax^m y^n)(bx^p y^q) = a\alpha^m \beta^n(b)x^{m+p}y^{n+q}$ for any $m, n, p, q \in \mathbb{N} \cup \{0\}$ (resp. $m, n, p, q \in \mathbb{Z}$).

Corollary 6. *Let R be a ring satisfying descending chain condition on right annihilators, α and β be ring automorphisms of R such that $\alpha\beta = \beta\alpha$. Then the following conditions are equivalent:*

- (1) $R[[x, y; \alpha, \beta]]$ (resp. $R[[x, y, x^{-1}, y^{-1}; \alpha, \beta]]$) is a left APP-ring.
- (2) For any countable subset A of R , $l_R(\sum_{a \in A} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} R\alpha^i \beta^j(a))$ (resp. $l_R(\sum_{a \in A} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} R\alpha^i \beta^j(a))$) is right s -unital.

If S the multiplicative monoid (\mathbb{N}, \cdot) , endowed with the usual order \leq , then $[[R^{(\mathbb{N}, \cdot), \leq}]]$ is the ring of arithmetical functions with values in R , endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d)g(n/d), \quad \text{for each } n \geq 1.$$

Corollary 7. *Let R be a ring satisfying descending chain condition on right annihilators. Then the following conditions are equivalent:*

- (1) $[[R^{(\mathbb{N}, \cdot), \leq}]]$ is a left APP-ring.
- (2) For any countable subset A of R , $l_R(\sum_{a \in A} Ra)$ is right s -unital.

Let (S, \leq) be a strictly totally ordered monoid which is also artinian. Then the set $X_s = \{(u, v) | u + v = s, u, v \in S\}$ is finite for any $s \in S$. Let V be a free Abelian additive group with the base consisting of elements of S . It was noted in [13] that V is a coalgebra over \mathbb{Z} with the comultiplication map and the counit map as follows:

$$\Delta(s) = \sum_{(u,v) \in X_s} u \otimes v, \quad \epsilon(s) = \begin{cases} 1, & s = 0, \\ 0, & s \neq 0, \end{cases}$$

and $[[R^{S, \leq}]] \cong \text{Hom}(V, R)$, the dual algebra with multiplication

$$f * g = (f \otimes g) \Delta \quad \forall f, g \in \text{Hom}(V, R).$$

Corollary 8. *Let (S, \leq) be a strictly totally ordered monoid which is also artinian, R a ring satisfying descending chain condition on right annihilators and $\text{Hom}(V, R)$ defined as above. Then the following conditions are equivalent:*

- (1) $\text{Hom}(V, R)$ is a left APP-ring.
- (2) For any S -indexed subset A of R , $l_R(\sum_{a \in A} Ra)$ is right s -unital.

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References

- [1] H. Al-Ezeh, Two properties of the power series ring, *Int. J. Math. Sci.* **11** (1988), 9-14.
- [2] E.P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.* **18** (1974), 470-473.
- [3] G.F. Birkenmeier, J.Y. Kim and J.K. Park, On quasi-Baer rings, *Contemp. Math.* **259** (2000), 67-92.
- [4] G.F. Birkenmeier, J.Y. Kim and J.K. Park, On polynomial extensions of principally quasi-Baer rings, *Kyungpook Math. J.* **40** (2000), 247-254.
- [5] G.F. Birkenmeier, J.Y. Kim and J.K. Park, Polynomial extensions of Baer and quasi-Baer rings, *J. Pure Appl. Algebra* **159** (2001), 25-42.
- [6] G.F. Birkenmeier, J.Y. Kim and J.K. Park, Principally quasi-Baer rings, *Comm. Algebra* **29** (2001), 639-660.
- [7] G.F. Birkenmeier and J.K. Park, Triangular matrix representations of ring extensions, *J. Algebra* **265** (2003), 457-477.
- [8] J.A. Fraser and W.K. Nicholson, Reduced PP-rings, *Math. Japon.* **34** (1989), 715-725.

- [9] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra* **168** (2002), 45-52.
- [10] C.Y. Hong, N.K. Kim and T.K. Kwak, Ore extensions of Baer and P.P.-rings, *J. Pure Appl. Algebra* **151** (2000), 215-226.
- [11] H. Kim, On t -closedness of generalized power series rings, *J. Pure Appl. Algebra* **166** (2002), 277-284.
- [12] H. Kim and T. I. Kwon, PF-rings of generalized power series, *Kyungpook Math. J.* **47** (2007), 127-132.
- [13] Z.K. Liu, Endomorphism rings of modules of generalized inverse polynomials, *Comm. Algebra* **28** (2000), 803-814.
- [14] Z.K. Liu and J. Ahsan, PP-rings of generalized power series, *Acta Math. Sinica, English Series* **16** (2000), 573-578.
- [15] Z.K. Liu, A note on principally quasi-Baer rings, *Comm. Algebra* **30** (2002), 3885-3890.
- [16] Z.K. Liu, Baer rings of generalized power series, *Glasgow Math. J.* **44** (2002), 463-469.
- [17] Z.K. Liu, Quasi-Baer rings of generalized power series, *Chinese Annals Math.* **23** (2002), 579-584.
- [18] Z.K. Liu and R.Y. Zhao, A generalization of PP-rings and p.q.-Baer rings, *Glasgow Math. J.* **48**(2006), 217-229.
- [19] Z.K. Liu, Triangular matrix representations of rings of generalized power series, *Acta. Math. Sinica, English Series* **22(4)** (2006) 989-998.
- [20] Z.K. Liu and X.Y. Yang, Left APP-property of formal power series, *Arch. Math. (Brno)* **44** (2008) 185-189.
- [21] Z.K. Liu and X.Y. Yang, On annihilators ideals of skew monoid rings, *Glasgow Math. J.* **52** (2010), 161-168.
- [22] G. Marks, R. Mazurk and M. Ziemkowski, A unified approach to various generalizations of Amendariz rings, *Bull. Austral. Math. Soc.* **81** (2010), 361-397.
- [23] R. Mazurk and M. Ziemkowski, On Bezout and distributive generalized power series rings, *J. Algebra* **306(2)** (2006), 397-411.
- [24] R. Mazurk and M. Ziemkowski, Uniserial rings of skew generalized power series, *J. Algebra* **318** (2007), 737-764.
- [25] R. Mazurk and M. Ziemkowski, On von Neumann regular rings of skew generalized power series, *Comm. Algebra* **36(5)** (2008), 1855-1868.

- [26] P. Ribenboim, Noetherian rings of generalized power series, *J. Pure Appl. Algebra* **79** (1992), 293-312.
- [27] P. Ribenboim, Special properties of generalized power series, *J. Algebra* **173** (1995), 566-586.
- [28] P. Ribenboim, Semisimple rings and von Neumann regular rings of generalized power series, *J. Algebra* **198** (1997), 327-338.
- [29] B. Stenström, *Rings of Quotients*, (Springer-Verlag, New York, 1975).
- [30] H. Tominaga, On s -unital rings, *Math. J. Okayama Univ.* **18** (1976), 117-134.
- [31] R.Y. Zhao and Z.K. Liu, Special properties of modules of generalized power series, *Taiwanese J. Math.* **12** (2008) 447-461.
- [32] R.Y. Zhao and Y.J. Jiao, Principal quasi-Baerness of modules of generalized power series, *Taiwanese J. Math.* accepted.